

DISTINGUISHING HECKE EIGENFORMS

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1. INTRODUCTION

Fourier expansions are a popular way of representing modular forms: they are explicit and easy to manipulate, and their coefficients often have arithmetic or combinatorial meaning and can therefore be interesting in their own right. For many applications, an essential question is: How many coefficients are sufficient in order to determine a modular form? This was answered first in the context of congruences modulo a prime by Sturm [10]. Let f and g be modular forms of the same weight k and level $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. Let \mathcal{O} be the ring of integers of the number field containing the coefficients of f and g , and let \mathfrak{p} be a prime ideal of \mathcal{O} . Sturm's result is that if $f \not\equiv g \pmod{\mathfrak{p}}$, then

$$\text{there exists } n \leq \frac{k}{12} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma] \quad \text{such that} \quad a_n(f) \not\equiv a_n(g) \pmod{\mathfrak{p}}.$$

This result was extended by Ram Murty [8] in several directions: working with forms of distinct levels and weights, replacing the “finite primes” with the “infinite prime”, and considering the important special case of Hecke newforms. Our interest lies with Theorem 4 in [8]: let f , resp. g be newforms of distincts weights on $\Gamma_0(N_1)$, resp. $\Gamma_0(N_2)$. Let $N = \mathrm{lcm}(N_1, N_2)$, then

$$\text{there exists } n \leq 4 \log^2(N) \quad \text{such that} \quad a_n(f) \neq a_n(g).$$

Murty gives a very elegant proof of this result. Unfortunately, the key estimate in the proof only holds for N large enough — it will follow from the proof of Lemma 3 in the next section that the estimate fails for N in the set

$$\{1, \dots, 4, 6, \dots, 12, 30, \dots, 33, 210, \dots, 244\}.$$

The statement itself is trivially false for $N = 1, 2$.

In this paper, we follow Murty's approach to prove the following statement:

Theorem 1. *Let f and g be cuspidal eigenforms of weights $k_1 \neq k_2$ on the group $\Gamma_0(N)$. Then*

$$(1) \quad \text{there exists } n \leq 4(\log(N) + 1)^2 \quad \text{such that} \quad a_n(f) \neq a_n(g).$$

We also indicate some better asymptotic bounds in (1) that follow from this method of proof, and we conclude with a section describing a computational experiment that tests how sharp the bounds are in the case of forms of level 1.

Conventions. We assume that any cuspidal eigenform f has been normalised so that the coefficient $a_1(f) = 1$. We use the standard notation p_k for the k -th smallest prime number.

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2. THE PROOF OF THEOREM 1

The starting point is the following result extracted from the proof of Theorem 4 in [8]:

Lemma 2 (Ram Murty). *Let f and g be cuspidal eigenforms of weights $k_1 \neq k_2$ on the group $\Gamma_0(N)$, and let p be the smallest prime not dividing N . Then*

$$\text{there exists } n \leq p^2 \text{ such that } a_n(f) \neq a_n(g).$$

Proof. We proceed by contradiction: suppose $a_n(f) = a_n(g)$ for all $n \leq p^2$. We have the well-known recurrence relation for Hecke operators on $\Gamma_0(N)$ and in weight k :

$$T_{p^2} = T_p^2 - p^{k-1} \langle p \rangle,$$

where $\langle p \rangle$ is the trivial character on $(\mathbb{Z}/N\mathbb{Z})^\times$, extended by zero to all of $\mathbb{Z}/N\mathbb{Z}$. Since eigenforms are normalised so $a_1 = 1$, the relation between Fourier coefficients and Hecke eigenvalues gives

$$\begin{aligned} a_{p^2}(f) &= a_p^2(f) - p^{k_1-1} \\ a_{p^2}(g) &= a_p^2(g) - p^{k_2-1}. \end{aligned}$$

By assumption we have $a_{p^2}(f) = a_{p^2}(g)$ and $a_p(f) = a_p(g)$, from which we derive $k_1 = k_2$, contradicting the hypothesis of the Lemma. \square

Proof of Theorem 1. According to Lemma 2, it is enough to show that for any $N \geq 1$ there exists a prime $p \leq 2 \log(N) + 2$ that does not divide N . Once again we proceed by contradiction: suppose N is divisible by all primes up to $2 \log(N) + 2$. Then

$$N \geq \prod_{p \leq 2 \log(N) + 2} p.$$

Using Chebyshev's function

$$\theta(x) = \sum_{p \leq x} \log(p),$$

we can rewrite the previous inequality as

$$\log(N) \geq \sum_{p \leq 2 \log(N) + 2} \log(p) = \theta(2 \log(N) + 2).$$

It will follow from Lemma 3 that the right hand side of this inequality is $> \log(N)$ for all $N \geq 1$, which leads to a contradiction. \square

Lemma 3. *Chebyshev's function satisfies*

$$\theta(2x + 2) > x \text{ for all } x \geq 0.$$

This estimate is an application of Theorem 1.4 in [4]:

Theorem 4 (Dusart). *Chebyshev's function satisfies*

$$|\theta(x) - x| < 3.965 \frac{x}{\log^2(x)} \text{ for all } x > 1.$$

It is worth noting that Dusart's results are based on detailed knowledge of the positions of the first 1.5×10^9 zeros of the Riemann zeta function, obtained numerically by Brent, van de Lune, te Riele, and Winter [12].

Proof of Lemma 3. We start by showing that

$$\theta(2x) > x \quad \text{for all } x > 8.356.$$

Indeed, suppose $x > 8.356$, then we have

$$x > \frac{1}{2} \exp(\sqrt{2 \cdot 3.965}) \Leftrightarrow \log^2(2x) > 2 \cdot 3.965.$$

By Dusart's estimate, we get

$$\theta(2x) - 2x > -3.965 \frac{2x}{\log^2(2x)} > -x.$$

Since θ is a step function, it is easy to check which values of $x \in [0, 8.356]$ do not satisfy the inequality $\theta(2x) > x$, namely

$$x \in [0, 3/2) \cup [\log 6, 5/2) \cup [\log 30, 7/2) \cup [\log 210, 11/2).$$

The discrepancy is largest at the right ends of the intervals, as can be seen in the Figure. We also notice that translating $\theta(2x)$ by -1 along the x -axis will disentangle the two graphs, in other words

$$\theta(2x + 2) > x \quad \text{for all } x \geq 0,$$

as claimed. □

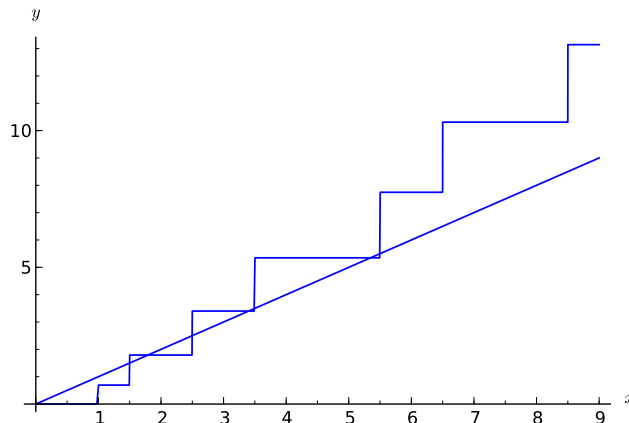


FIGURE 1. Graphs of $y = \theta(2x)$ and $y = x$

3. ASYMPTOTIC UPPER BOUNDS

The upper bound appearing in the statement of Theorem 1 was chosen because it is a very simple function and because it holds for all values of the level N . However, the reader will have realised from the estimates we used that this bound gets less and less sharp as N increases (because $\theta(x) \sim x$ by the Prime Number Theorem). We use some known results on the behaviour of the prime gaps $g_k = p_{k+1} - p_k$ to give better unconditional and conditional asymptotic upper bounds.

Theorem 5. *Let f and g be cuspidal eigenforms of weights $k_1 \neq k_2$ on the group $\Gamma_0(N)$. Then*

(1) *there exists*

$$n = O\left((\log(N) + \log(N)^{0.525})^2\right)$$

such that $a_n(f) \neq a_n(g)$;

(2) *assuming the Riemann hypothesis, there exists*

$$n = O\left((\log(N) + \log(N)^{0.5} \log \log(N))^2\right)$$

such that $a_n(f) \neq a_n(g)$;

(3) *assuming Cramér's conjecture on prime gaps (see [6]), there exists*

$$n = O\left((\log(N) + (\log \log(N))^2)^2\right)$$

such that $a_n(f) \neq a_n(g)$.

Proof. The key point is to estimate the size of the smallest prime $p(N)$ not dividing N in terms of N . Since we are looking for an upper bound, we are naturally led to focus on the worst-case scenario, the *primorials*

$$N_k = p_1 p_2 \dots p_k = \exp(\theta(p_k)),$$

for which we clearly have $p(N_k) = p_{k+1}$. Writing $g_k = p_{k+1} - p_k$ for the gap between consecutive primes, we have

$$p_{k+1} = p_k + g_k = \begin{cases} p_k + O(p_k^{0.525}) & \text{unconditional, see Baker-Harman-Pintz [1]} \\ p_k + O(\sqrt{p_k} \log(p_k)) & \text{assuming RH, see Cramér [3]} \\ p_k + O(\log^2(p_k)) & \text{assuming Cramér's conjecture, see [6].} \end{cases}$$

Simple manipulations together with the fact that $p_k \sim \theta(p_k) = \log(N_k)$ give us the upper bounds in the statement. \square

4. A NUMERICAL EXPERIMENT

Since the results in the previous section all build upon Murty's approach in Lemma 2, it is natural to ask how tight the bound of Lemma 2 is.

Let us consider the level 1 case. Here, Lemma 2 tells us that there exists $n \leq 4$ such that $a_n(f) \neq a_n(g)$. Is it possible to improve on the 4? We investigated this question via a computational approach, which we will describe after we recall the following

Conjecture (Maeda, Conj. 1.2 in [7]). *The characteristic polynomial of the Hecke operator T_2 acting on the space of cusp forms $S_k(\mathrm{SL}_2(\mathbb{Z}))$ is irreducible.*

Maeda's conjecture has been verified numerically by Farmer-James [5], Buzzard [2], Stein, and Kleinerman¹ for all weights ≤ 3000 , except for 2796. We have verified the case $k = 2796$ using the mathematical software Sage [9] (the computation of the characteristic polynomial used native Sage code, and the check for irreducibility used polynomial factorisation code from PARI/GP [11]).

Based on a sample of results for small weights, our project was to compute the Fourier coefficient a_2 of all cuspidal eigenforms of level 1 and weights ≤ 10000 . To each a_2 we associate its characteristic polynomial over \mathbb{Q} . If we assume Maeda's conjecture, then if we want to

¹See http://wstein.org/Tables/charpoly_level1/t2/

detect duplicates in the list of a_2 's, it suffices to look for duplicates in the list of characteristic polynomials. To make our search even more efficient, instead of computing (and storing) the characteristic polynomial corresponding to each weight k , we simply compute and store the degree and the trace of T_2 on the space $S_k(\mathrm{SL}_2(\mathbb{Z}))$. This reduces the computations that we need to perform to the following:

- (1) Find the Victor Miller basis of $S_k(\mathrm{SL}_2(\mathbb{Z}))$

$$\begin{aligned} q &+ \dots =: f_1(q) \\ q^2 &+ \dots =: f_2(q) \\ &\vdots \\ q^d &+ \dots =: f_d(q), \end{aligned}$$

where $d = d_k = \dim S_k(\mathrm{SL}_2(\mathbb{Z}))$ and each q -expansion is computed up to and including the coefficient of q^{2d} (the precision required for computing the action of T_2 in the next step).

- (2) For $j = 1, \dots, d$, compute the coefficient of q^j in $T_2 f_j$, given by $a_{2j}(f_j) + 2^{k-1} a_{j/2}(f_j)$.
- (3) The trace t_k of T_2 is the sum of the coefficients computed in the previous step. Store the pair (d_k, t_k) .

This algorithm was implemented in Sage [9] and run in parallel (one instance per value of k) on Linux servers {cerelia, skadi, soleil}.ms.unimelb.edu.au at the University of Melbourne, and {geom, mod, sage}.math.washington.edu at the University of Washington².

After running³ this algorithm over the range of weights $2 \leq k \leq 10000$, we found that the list of pairs (d_k, t_k) contained no duplicates. We record this result as

Theorem 6. *Comparing the Fourier coefficient a_2 is sufficient to distinguish all cuspidal eigenforms of level 1 and weights ≤ 3000 . If we assume Maeda's conjecture, the same is true for weights ≤ 10000 .*

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³The instance $k = 10000$ required about 50 minutes on one core of a Quad-Core AMD Opteron 8356 processor, and 5.4Gb of memory.

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